Impulsive Shells of Null Dust Colliding with Gravitational Plane Waves

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A two-parameter family of solutions of Einstein's equations, correspond-
ing to distribution valued stress-energy tensors with support on a (pair
of intersecting) null hypersurface(s), is presented. They describe the col-
lision of infinitely thin shells of null dust colliding with shells of the same
kind and/or gravitational plane waves. For a subclass of this new family
of solutions, the typical spacelike singularity that develops after the col-
lision and forms the future boundary of the interaction region gives its
place to a nonsingular Killing-Cauchy horizon.

1. INTRODUCTION

The process of collision between shells of null dust (a gas of massless par-
ticles which follow a congruence of null trajectories) has attracted a lot
of interest recently [1-8], especially after the discovery of Chandrasekhar
and Xanthopoulos [1] that the product of such a kind of collision may
be a shell of "stiff matter", i.e. a perfect fluid with an equation of state
p (pressure) $= \mu$ (energy density).

The case of a pair of impulsive (infinitely thin) shells of null dust
colliding with each other was first considered by Dray and 't Hooft [4].
The exact model constructed by them showed that the above process has
several features in common with the collision of a pair of impulsive grav-
itational plane waves. Each pulse acts as a focusing lense for the rays

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corresponding to the other; a Coulomb-like gravitational field appears in the "region of interaction"; a space-time singularity is formed along a spacelike hypersurface which forms the future boundary of the region of interaction.

The equally interesting process of an impulsive shell of null dust colliding with an impulsive gravitational plane wave was considered by Babala [9]. Tsoubelis [10], on the other hand, showed that the Dray-'t Hooft and Babala models referred to above form the first two members of a class of solutions which can be obtained by choosing appropriately the values of the parameters which characterize the well known Szekeres [11] family of "colliding plane gravitational waves" solutions. The rest of the solutions obtained in [10], which will be referred to hereafter as Paper I, represent the collision of an impulsive plane shell of null dust with a plane gravitational wave having different types of profile such as shock, with a smooth wavefront etc.

In the present paper we extend the results of Paper I in the following sense. Using the three-parameter family of the Einstein vacuum equations obtained by the present authors recently [12], we derive a one-parameter generalization of the family of solutions with distribution valued stress-energy tensor having its support on a (pair of intersecting) null hypersurface(s), which was obtained in Paper I. This generalization leads to a two-parameter class of models, the general member of which represents the following physical process. An impulsive plane shell of null dust, riding a plane gravitational wave which has both an impulsive and a shock component, propagates in a flat region of space-time and, eventually collides with either a similar type of shell or a gravitational wave incident from the opposite direction. However, the particular models that obtain by tuning the two free parameters appearing in the general solution cover a wide range of physically distinct cases, such as shell–shell, shell–impulsive wave, (shell+impulsive wave)–(shell+impulsive wave) collision. This allows for several aspects of the process of collision involving impulsive shells of null matter to come to the fore. Moreover, a one-parameter subfamily of solutions is obtained in which no space-time singularity appears along the "focusing hypersurface". Thus, a closer relation is established between collision involving impulsive shells of null dust, on the one hand, and collision between gravitational waves, on the other, because, for the latter type of collisions, the existence of solutions which do not develop a spacelike singularity in the region of interaction is already well established [12-17].

The paper is structured as follows. In Section 2, we present the basic geometric features (metric, Ricci tensor, Weyl scalars) of the space-time models employed in our later construction. This is followed by a short

presentation of the Tsoubelis-Wang family of vacuum solutions mentioned above, as well as of the manner in which the Khan-Penrose [18] extension algorithm leads to the appearance of null matter sheets in the space-time manifold. In Section 3, we present the new two-parameter family of models which represent collisions involving impulsive shells of null dust and gravitational plane waves. The paper closes with a detailed analysis of the physical features and singularity structure of a series of representative models which are obtained by assigning a particular set of values to the pair of parameters characterizing the above family of solutions.

2. THE FIELD EQUATIONS AND THEIR SOLUTION

For the space-time models to be considered in the following one can find a coordinate system $x_\lambda = (x^0, x^1, x^2, x^3) = (u, v, x, y)$ in which the metric takes the form

$$ds^2 = 2e^{-M} du dv - e^{-U}(e^V dx^2 + e^{-V} dy^2) \qquad (1)$$

where M, U and V are functions of the null coordinates u and v only. This means that the models under consideration have a hypersurface orthogonal pair of spacelike Killing vector fields, $(\vec{\partial}_x, \vec{\partial}_y)$.

The nonvanishing components of the Ricci tensor $R_{\lambda\mu}$ corresponding to the above metric are given by

$$
\begin{aligned}
R_{uu} &= -\tfrac{1}{2}(2U_{,uu} - U_{,u}^2 + 2U_{,u}M_{,u} - V_{,u}^2) \\
R_{vv} &= -\tfrac{1}{2}(2U_{,vv} - U_{,v}^2 + 2U_{,v}M_{,v} - V_{,v}^2) \\
R_{uv} &= R_{vu} = -\tfrac{1}{2}(2M_{,uv} + 2U_{,uv} - U_{,u}U_{,v} - V_{,u}V_{,v}) \\
R_{xx} &= \tfrac{1}{2}e^{M-U+V}(2U_{,uv} - 2V_{,uv} - 2U_{,u}U_{,v} + U_{,u}V_{,v} + U_{,v}V_{,u}) \\
R_{yy} &= \tfrac{1}{2}e^{M-U-V}(2U_{,uv} + 2V_{,uv} - 2U_{,u}U_{,v} - U_{,u}V_{,v} - U_{,v}V_{,u}) \quad (2)
\end{aligned}
$$

where $U_{,u} \equiv \partial U/\partial u$, $U_{,uv} \equiv \partial^2 U/\partial u \partial v$, etc.

A convenient expression of the conformal or Weyl curvature tensor $C_{\alpha\beta\gamma\delta}$ corresponding to (1) is provided by the Weyl scalars Ψ_A, $A = 0, 1, 2, 3, 4$. In terms of the null tetrad $(\ell^\lambda, n^\lambda, m^\lambda, \bar{m}^\lambda)$, where

$$\ell^\lambda = e^{M/2}\delta_1^\lambda, \qquad n^\lambda = e^{M/2}\delta_0^\lambda,$$

$$m^\lambda = \frac{1}{\sqrt{2}}e^{(U-V)/2}(\delta_2^\lambda + ie^V \delta_3^\lambda),$$

$$\bar{m}^\lambda = \frac{1}{\sqrt{2}}e^{(U-V)/2}(\delta_2^\lambda - ie^V \delta_3^\lambda), \qquad (3)$$

the Weyl scalars are given by

$$\Psi_0 \equiv -C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta = -\tfrac{1}{2}e^M[V_{,vv} + (M_{,v} - U_{,v})V_{,v}]$$
$$\Psi_1 \equiv -C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \ell^\gamma m^\delta = 0$$
$$\Psi_2 \equiv -\tfrac{1}{2}C_{\alpha\beta\gamma\delta}(\ell^\alpha n^\beta \ell^\gamma n^\delta - \ell^\alpha n^\beta m^\gamma \bar{m}^\delta) = \tfrac{1}{6}e^M(M_{,uv} - U_{,uv} + V_{,u}V_{,v})$$
$$\Psi_3 \equiv \ \ C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta n^\gamma \bar{m}^\delta = 0$$
$$\Psi_4 \equiv -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta = -\tfrac{1}{2}e^M[V_{,uu} + (M_{,u} - U_{,u})V_{,u}] \tag{4}$$

Now let (a, k_1, k_2) be a triad of real parameters, the range of the last two being restricted by the conditions

$$2m \equiv \frac{2}{2 - (k_1 - k_2)^2} \geq 1 \quad \text{and} \quad 2n \equiv \frac{2}{2 - (k_1 + k_2)^2} \geq 1 \tag{5}$$

Furthermore, let

$$\begin{aligned} b &\equiv (a + k_1)^2 - \tfrac{1}{4}, & c &\equiv (a - k_1)^2 - \tfrac{1}{4} \\ \beta &\equiv (a + k_2)^2 - \tfrac{1}{4}, & \gamma &\equiv (a - k_2)^2 - \tfrac{1}{4}. \end{aligned} \tag{6}$$

Then, the expressions

$$e^{-M} = R^{-2+1/n}S^{-2+1/m}(1 - \eta)^b(1 + \eta)^c(1 - \mu)^\beta(1 + \mu)^\gamma,$$
$$e^{-U} = T,$$
$$e^V = T^{2a}\left(\frac{1 - \eta}{1 + \eta}\right)^{k_1}\left(\frac{1 - \mu}{1 + \mu}\right)^{k_2}, \tag{7}$$

where

$$\begin{aligned} R &\equiv \sqrt{1 - v^{2m}}, & S &\equiv \sqrt{1 - u^{2n}}, \\ \eta &\equiv u^n R + v^m S, & \mu &\equiv u^n R - v^m S, \\ T &\equiv \sqrt{(1 - \eta^2)(1 - \mu^2)} = 1 - u^{2n} - v^{2m}, \end{aligned} \tag{8}$$

provide a three-parameter family of exact solutions of the vacuum field equations $R_{\lambda\mu} = 0$. This was shown in [12], which will be referred to hereafter as Paper II. (Please note a slight change of notation: the parameters δ_i appearing in Paper II are denoted by k_i in the present paper). On the other hand, the metric defined by eqs. (7) above is an explicit version of the solution which results from setting $A_\omega = 0$, $d_i = 0$ for $i > 2$, in eq. (13) of [17].

As it stands, the solution given by eqs. (5)–(8) is valid only in Region IV of Fig. 1, i.e. in the interior of that region of space-time which is bounded by the hypersurfaces $u = 0$, $v = 0$ and $T = 0$. However, an extension of the above solution toward the past of Region IV can be immediately obtained using a) the assumption $R_{\lambda\mu} = 0$ in the region of interaction and b) the technique of Khan and Penrose [18]. The latter consists of letting

$$u \to uH(u) \qquad \text{and} \qquad v \to vH(v), \tag{9}$$

H being the Heaviside unit step function, in the metric coefficients given by (7).

The fact that the vacuum field equations are also satisfied in Regions I-III of Fig. 1, as a result of the above substitution, is easily verified. However, this is not the case, in general, along the hypersurfaces $u = 0$ and $v = 0$ separating Regions I-IV from each other.

In order to see how this comes about in the general case, let us return to (1) and make the following assumptions. The metric coefficients given by (1) have finite limits as $u \to 0^+$ and/or $v \to 0^+$ and, after making the substitutions (9), these coefficients satisfy the vacuum field equations on both sides of the hypersurfaces $u = 0$ and $v = 0$. Then, it follows from eqs. (2) that

$$R_{uu} = -\frac{\partial U(u, vH(v))}{\partial u}\,\delta(u), \qquad R_{vv} = -\frac{\partial U(uH(u), v)}{\partial v}\,\delta(v), \tag{10}$$

where δ denotes Dirac's delta function, and the rest of the components of the Ricci tensor vanish everywhere. Combining (10) with Einstein's field equations, we conclude that, in general, the stress energy tensor $T_{\lambda\mu}$ does not vanish along the hypersurfaces $u = 0$ and $v = 0$.

Returning to the solution given by (5)–(8), we find that the field equations and (10) imply that

$$\kappa T_{uu} = -R_{uu} = \frac{2n\,u^{2n-1}\delta(u)}{1 - v^{2m}H(v)} \tag{11}$$

and

$$\kappa T_{vv} = -R_{vv} = \frac{2m\,v^{2m-1}\delta(v)}{1 - u^{2n}H(u)} \tag{12}$$

where $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant. Therefore, when $2m \neq 1$ and $2n \neq 1$, the separation hypersurfaces $u = 0$ and $v = 0$ are matter-free and, as shown in Paper II, the corresponding models admit the following interpretation.

Two pulses of plane gravitational waves propagate toward each other in Region I, which is flat. The pulses have, in general, different profiles and their leading edges are represented by the null hypersurfaces $u = 0$ and $v = 0$, respectively. At $(u, v) = (0, 0)$ the waves collide and the outcome of this collision is reflected in the structure of Region IV, the "region of interaction." In general, the mutual focusing of the wave pulses leads to the development of a space-time singularity along the "focusing hypersurface" $T = 0$ which forms the future boundary of Region IV.

If, on the other hand, either $2m$ or $2n$ equals unity, then at least one of the separation hypersurfaces is occupied by matter in the form of "null dust." This being the case of interest in the present paper, we turn to an analysis of the subfamily of models for which

$$2n = 1 \longleftrightarrow k_1 = -k_2 \equiv k. \tag{13}$$

2. COLLISIONS INVOLVING IMPULSIVE PLANE SHELLS OF NULL DUST AND GRAVITATIONAL PLANE WAVES

Combining (11) and (13) we find that

$$\kappa T_{uu} = \frac{\delta(u)}{1 - v^{2m}H(v)} \tag{14}$$

This means that, in all the space-time models to be considered in the following, an infinitely thin shell of null dust propagates along $u = 0$, i.e. toward the left in Fig. 1.

The combination of (4)–(8) and (13), on the other hand, leads to the following expressions for the nonvanishing Weyl scalars in Region IV:

$$\Psi_0^{IV}(u, v) = 8m^2 v^{m-2} \left\{ \frac{b(av^{3m} + kS^3)}{T^2} \right. \\ \left. - \frac{ka[(2a + k)v^m + (a + 2k)S]}{(v^m + S)^2} \right\} \tag{15}$$

$$\Psi_2^{IV}(u, v) = \frac{2bmv^{2m-1}}{T^2} + \frac{2kamv^{m-1}}{S(v^m + S)^2} \tag{16}$$

$$\Psi_4^{IV}(u, v) = \frac{2b(kv^{3m} + aS^3)}{S^3T^2} - \frac{2ka[(2a + k)S + (a + 2k)v^m]}{S^3(v^m + S)^2} \tag{17}$$

In writing down the above expressions for Ψ_A, a factor of $\exp(M)$ was omitted. This is equivalent to replacing the Weyl scalars Ψ_A with Szekeres'
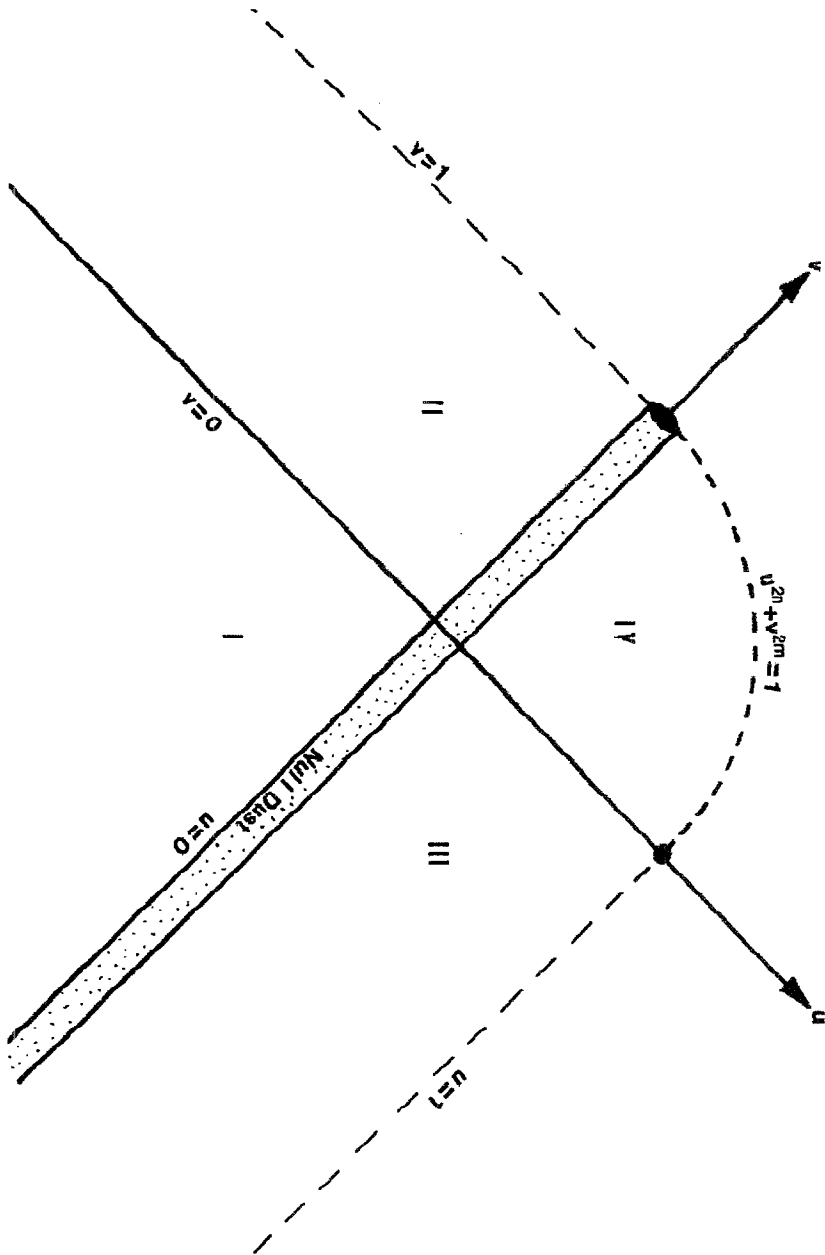
Figure 1 The (u, v)-plane of the space-time models described in the test.
An infinitely thin shell of null dust, accompanied in some models by gravitational radiation, is incident from the right along $u = 0$. At $(u, v) = (0, 0)$ it collides with a similar kind of shell, or a gravitational plane wave, incident from the left along $v = 0$.

[11] "scale invariant Weyl scalars" $\Psi_A{}^0$, because $\Psi_A{}^0 = \exp(-M)\Psi_A$, in the present case. The above practice will be maintained in the following discussion, and so the term "Weyl scalars" will refer to the $\Psi_A{}^0$'s.

On the basis of (15)–(17) we can, firstly, make the following general statement. As $T \to 0^+$, while $u \neq 0$ and $v \neq 0$, a curvature singularity develops in all of our models, except in those for which

$$b = 0 \longleftrightarrow a + k = \pm 1/2. \tag{18}$$

In the latter case, one of the Killing vectors $\vec{\partial}_x$, $\vec{\partial}_y$ becomes null as $T \to 0^+$ (equivalently, as $\eta \to 1^-$), as can be seen by combining (7), (8), (13) and (18). Thus, when (18) holds, the space-like singularity that otherwise develops along $T = 0$ gives its place to a Killing-Cauchy horizon beyond which the metric can be analytically extended.

Moreover, using (15)–(17) and the Khan-Penrose substitutions (9) we can immediately obtain the Weyl scalars in Regions I-III. They are given by

$$\Psi_A^I = 0, \tag{19}$$

$$\Psi_0^{II}(v) = \frac{2m}{(1 - v^{2m})^2} [ma(4a^2 - 1)v^{4m-2} + 12ka^2mv^{3m-2}$$

$$+ 3a(2m - 1)v^{2m-2} + k(m - 1)v^{m-2}] \tag{20}$$

$$\Psi_A^{II} = 0, \qquad \text{when} \quad A \neq 0, \tag{21}$$

$$\Psi_4^{III}(u) = \frac{a(4a^2 - 1)}{2(1 - u)^2} \quad \text{and} \quad \Psi_A^{III} = 0, \qquad \text{when} \quad A \neq 4. \tag{22}$$

In order to determine the Weyl scalars in the neighborhood of $u = 0$ or $v = 0$, on the other hand, we must return to (4). Using this equation and (9), we find the following behavior in the neighborhood of that portion of the $u = 0$ or $v = 0$ hypersurface along which the pair of regions listed on the first column meet.

$$I - III: \quad \Psi_4^{I-III} = H(u)\Psi_4^{III}(o) + a\delta(u) \tag{23}$$

$$II - IV: \quad \Psi_4^{II-IV} = H(u)\Psi_4^{IV}(o,v) + \frac{a + kv^m}{1 - v^{2m}}\delta(u) \tag{24}$$

$$\Psi_2^{II-IV} = H(u)\Psi_2^{IV}(o,v)$$

$$\text{and } \Psi_0 \text{ continuous} \tag{25}$$

$$I - II: \quad \Psi_0^{I-II} = H(v)\Psi_0^{II}(o)$$

$$+ 2m(kv^{m-1} + av^{2m-1})\delta(v) \qquad (26)$$

$$III - IV : \quad \Psi_0^{III-IV} = H(v)\Psi_0^{IV}(u, o)$$

$$+ \frac{2m(kv^{m-1}S + av^{2m-1})}{1 - u}\delta(v) \qquad (27)$$

$$\Psi_2^{III-IV} = H(v)\Psi_2^{IV}(u, o)$$

and Ψ_4 continuous. \qquad (28)

At this point we have at our disposal all the quantities that are necessary for determining the physical character of our models. The latter, however, is brought out most clearly by considering the cases that correspond to distinct values of the parameter m separately, a task to which we now turn.

Case A: $m = 1/2$

When $m = 1/2(k = 0)$, (12) and (14) lead to

$$\kappa T_{vv} = \frac{\delta(v)}{1 - uH(u)}, \qquad \kappa T_{uu} = \frac{\delta(u)}{1 - vH(v)}, \qquad (29)$$

while from (15)–(28) we find that

$$\Psi_0 = \frac{a(4a^2 - 1)H(v)}{2[1 - u\,H(u) - v\,H(v)]^2} + \frac{a\delta(v)}{1 - u\,H(u)}, \qquad (30)$$

$$\Psi_4 = \frac{a(4a^2 - 1)H(u)}{2[1 - u\,H(u) - v\,H(v)]^2} + \frac{a\delta(u)}{1 - v\,H(v)}, \qquad (31)$$

and

$$\Psi_2 = \frac{(4a^2 - 1)H(u)H(v)}{4[1 - u\,H(u) - v\,H(v)]^2} \qquad (32)$$

It is clear from (29)–(32) that, provided $a \neq 0, \pm 1/2$, the $m = 1/2$ models represent the symmetric collision of a pair of impulsive shells of null dust each of which is accompanied by an impulsive–plus–shock gravitational plane wave. The same equations show that, following the collision, the null dust shells, as well as the gravitational waves, begin to focus and a Coulomb-like gravitational field (represented by Ψ_2) appears in Region IV. The strength of this field grows beyond all bounds as one approaches the spacelike hypersurface $u + v = 1$ and, as a result, a space-time singularity forms along the future boundary of the region of interaction.

Case A-1: $a = 0$.

In this case $\Psi_0 = \Psi_4 = 0$ and, therefore, the collision involves a pair of impulsive shells of null dust, only. The effects obtained following the collision are as described in the previous paragraph and have the same character as those that arise after the collision of a pair of impulsive gravitational waves [18]. The present subcase was first considered by Dray and 't Hooft in [4] where the reader can find more details regarding the singularity structure of the corresponding metric.

Case A-2: $a = \pm 1/2$.

Now, the null dust shells of the previous subcase are accompanied by impulsive gravitational waves. As seen from (30)–(32), the result of this superposition is that no Coulomb-like gravitational field appears in Region IV and no space-time singularity develops along the hypersurface $u+v = 1$. Space-time remains flat everywhere outside the null hypersurfaces $u = 0$ and $v = 0$. The only effect that obtains in this kind of collision is the mutual focusing of the participants.

The quenching of the Ψ_2 or Coulomb-like component observed in the present subcase illustrates the crucial role played by non-vanishing stress-energy sources in the outcome of a collision of plane gravitational waves. As shown by Szekeres [11], in the absence of such sources, the appearance of a Coulomb-like field is an invariable feature of the collision process and this holds even when the collision involves only a pair of impulsive waves, as illustrated by the well known solution of Khan and Penrose [18].

Before turning to the asymmetric solutions which correspond to $m > 1/2$, let us note that the totality of the Case A metrics were first obtained by Stoyanov [19]. Stoyanov, however, interpreted them as representing the collision of a pair of (empty) gravitational waves. That this interpretation cannot be supported was first noted by Nutku [20] and is made clear from the analysis presented above.

Case B: $m = 1$

When $m = 1$ ($k = \epsilon/2 \equiv \pm 1/2$), we find that

$$\kappa T_{uu} = \frac{\delta(u)}{1 - v^2 H(v)}, \qquad T_{vv} = 0 \tag{33}$$

and

$$\Psi_4^{I-III} = \tfrac{1}{2}a(4a^2 - 1)H(u) + a\delta(u), \tag{34}$$

$$\Psi_0^{I-II} = 6aH(v) + \epsilon\delta(v), \tag{35}$$

$$\Psi_4^{II-IV} = \frac{a(3v^2 + 6\epsilon av + 4a^2 - 1)}{2(1 - v^2)^2} H(u) + \frac{2a + \epsilon v}{2(1 - v^2)} \delta(u), \tag{36}$$

$$\Psi_0^{III-IV} = \frac{6aH(v)}{1-u} + \frac{\epsilon\delta(v)}{\sqrt{1-u}}. \tag{37}$$

From (33)–(35), it is clearly seen that the general Case B model represents the same type of collision as the typical Case A model, except for the fact that, now, the gravitational wave incident from the left is not accompanied by a shell of null dust. According to (33), (36) and (37), on the other hand, both the null dust shell and the gravitational waves involved in the collision end up in a singularity which stretches all along the arc $u = 1 - v^2$, $0 \leq v \leq 1$, of the (u, v) plane, unless $a = 0, -\epsilon$ [cf. (18)].

Case B-1: $a = 0$.

In this case, (34) and (35) reduce to $\Psi_4^{I-III} = 0$ and $\Psi_0^{I-II} = \epsilon\delta(v)$, respectively. In fact, one easily finds that the only nonvanishing Weyl scalars are Ψ_0 and Ψ_4, which are given by

$$\Psi_0(u, v) = \frac{\epsilon\delta(v)}{\sqrt{1 - uH(u)}}, \qquad \Psi_4(u, v) = \frac{\epsilon v H(v)\delta(u)}{2(1 - v^2)}. \tag{38}$$

Therefore, this model represents the collision of an impulsive gravitational plane wave with a shell of null dust of the same type. Space-time is flat in all Regions I-IV and the only effect of the collision is the mutual focusing of the wave pulse and the null matter shell. The corresponding metric was first obtained by Babala [9], who also considered its extension beyond the $u = 1 - v^2$ hypersurface.

Case B-2: $a = \epsilon'/2 \equiv \pm 1/2$.

Now, (34) and (35) reduce to

$$\Psi_4^{I-III} = \tfrac{1}{2}\epsilon'\delta(u) \tag{39}$$

and

$$\Psi_0^{I-II} = 3\epsilon' H(v) + \epsilon\delta(v), \tag{40}$$

respectively. This means that the present solution corresponds to the collision of a shock wave with an impulsive shell of null dust both of which are acompanied by an impulsive gravitational wave. In this case, the formation of a space-time singularity along the hypersurface $u = 1 - v^2$ is not avoided.

Case B - 3: $a = -2k$.

Since in this case

$$\Psi_4^{I-III} = \tfrac{3}{2}\epsilon H(u) - \epsilon\delta(u) \tag{41}$$

and

$$\Psi_0^{I-II} = -6\epsilon H(v) + \epsilon\delta(v), \tag{42}$$

the corresponding collision is of the general Case B type described after (37). However, the outcome of the collision is not typical. The superposition of gravitational wave and null dust in each leg of the incoming radiation is such that the strength of the Coulomb-like gravitational field that develops in Region IV after the collision remains finite as one approaches the $v > 0$ part of the $u = 1 - v^2$ hypersurface. Specifically,

$$\Psi_2^{IV} = \frac{1}{\sqrt{1 - u}\,(v + \sqrt{1 - u}\,)^2}, \tag{43}$$

and, thus,

$$\Psi_2(1 - v^2, v) = -\frac{1}{4v^3}. \tag{44}$$

Therefore, no singularity appears toward the future of Region IV and the model can be extended analytically in this direction.

Case C: $m = 2$

When $m = 2$ $(k = \epsilon\sqrt{(3/8)}\,)$, we find that

$$\Psi_4^{I-III} = \tfrac{1}{2}a(4a^2 - 1)H(u) + a\delta(u) \tag{45}$$

and

$$\Psi_0^{I-II} = 4kH(v). \tag{46}$$

Thus, the typical Case C model represents the collision of a shock gravitational wave incident from the left with an impulsive shell of null dust which is incident from the right and is accompanied by a gravitational shock–plus–impulsive wave. The behavior of this subclass of models after collision is similar to the one obtained in cases A and B above, the main feature being the development of a space-like singularity along the $u = 1 - v^4$ arc of the (u, v) plane, unless $a = -k \pm 1/2$. Therefore, we will refrain from giving any further details about the $m = 2$ metrics except for mentioning that the $a = 0$ and $a = \pm 1/2$ models are physically distinct from their Case C partners. When $a = 0$, the null dust shell incident from the right is not accompanied by any gravitational wave and, when $a = \pm 1/2$, it is only an impulsive wave that accompanies the shell of null dust.

Case D: $m = 4$

When $m = 4$ $(k = \epsilon\sqrt{(7/16)}\,)$, the shock wave incident from the left in Case C is replaced by a gravitational plane wave with a smooth wavefront. This follows from the fact that, in this case,

$$\Psi_0^{I-II} = \Psi_0^{III-IV} = 0. \tag{47}$$

On the other hand,

$$\Psi_4^{I-III} = \tfrac{1}{2}a(4a^2 - 1)H(u) + a\delta(u), \tag{48}$$

as in the previous cases, and

$$\Psi_4^{II-IV} = \frac{a(4a^2 - 1) + 12a^2kv^4 + 12ak^2v^8 + k(4k^2 - 1)v^{12}}{2(1 - v^8)^2} H(u)$$
$$+ \frac{a + kv^4}{1 - v^8} \delta(u). \tag{49}$$

From (48) and (49) it is easily deduced that the behavior of the $m = 4$ models after collision follows the pattern described earlier, in connection with Cases A-C. However, it is worth pointing out that the present case illustrates in the clearest possible fashion the effect of secondary gravitational wave production which arises essentially in all the space-time models constructed in the present paper. Consider, in this direction, the $a = 0$ subcase. For this particular model $\Psi_4^{I-III} = 0$. Therefore, the null dust shell incident from the right is not accompanied by any gravitational radiation. According to (49), on the other hand,

$$\Psi_4^{II-IV} = \frac{3kv^{12}}{8(1 - v^8)^2} H(u) + \frac{kv^4}{1 - v^8} \delta(u). \tag{50}$$

This shows that, upon entering the body of the gravitational wave pulse incident from the left, the shell of null dust stimulates the emission of gravitational radiation in its own direction, besides becoming focused as it proceeds [cf. (14)].

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